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Mixed convection plume above a point heat source in a vertical free stream

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Abstract—The mixed convection boundary layer above a point heat source subjected to a vertically upward uniform stream is considered. In the present work a new approach is presented where entire mixed convection domain is described by a single set of equations. The numerical solutions are presented for Prandtl numbers of 0.72, 6.8, 50, 100 and infinity.

1. INTRODUCTION

THE MIXED convection boundary layer due to an axisymmetric buoyant plume subjected to a vertically directed uniform stream has been studied by Appalaswamy and Jaluria [1], Afzal [2] and Riley and Drake [3]. The two cases of aiding and opposing flows can arise if the buoyant force vector and the oncoming stream are, respectively, in the same and opposite directions. The complete solution for the entire mixed convection regime in both aiding and opposing flow situations has been given by Afzal [2] for Prandtl number $Pr = 0.72$.

In the opposing flow situation, where the buoyancy effect opposes the oncoming stream, the dual solutions with a turning point have been reported [2]. These dual solutions correspond to the forward and reverse flows in the plume. The work of Riley and Drake [3] presents the solutions for $Pr = 1$ for the forward flow domain for two specific values of mixed convection parameter. On the other hand the formulation of Applaswamy and Jaluria [1], for the opposing flow situation, is in error as these authors have employed the negative sign with the boundary condition on axial velocity at infinity rather than with the buoyancy term in the momentum equation [4].

For the aiding flow situation, where buoyancy effect assists the oncoming stream, the solution is unique [2]. Applaswamy and Jaluria [1] have studied the case of a strongly buoyant plume by considering small perturbations to a free convection plume for $Pr = 0.7$ and 7. The solutions covering the entire range from purely forced convection to a purely free convection buoyant plume has been given by Afzal [2] for $Pr = 0.72$ and Riley and Drake [3] for $Pr = 1$. The studies [2, 3] are based on two sets of equations: one set in forced convection dominated variables and the second set in free convection dominated variables with a change over at certain value of mixed convection parameter. In the present work a new formulation is presented that enables one to describe the entire mixed convection regime by a single set of equations. The solutions to new set of equations have been obtained for various values of $Pr = 0.72$, 6.8, 50, 100 and ∞.

2. ANALYSIS

The boundary-layer equations for an axisymmetric fluid motion under the Boussinesq approximation are

$$y\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}(yv) = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = g\beta^*(T-T_\infty)+\frac{\nu}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right) \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \frac{\nu}{y\,Pr}\frac{\partial}{\partial y}\left(y\frac{\partial T}{\partial y}\right). \tag{3}$$

The velocity and temperature distributions are symmetric with respect to the vertical axis and far away approach their free stream values

$$y = 0 \quad v = \frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = 0 \tag{4}$$

$$y \to \infty \quad u \to U_\infty \quad T \to T_\infty. \tag{5}$$

The integration of energy equation across the plume yields

$$2\pi\rho C_p \int_0^\infty u(T-T_\infty)y\,dy = Q \tag{6}$$

where Q is the rate of heat released from the point source.

In terms of similarity variable ζ, the stream function ψ and temperature T can be expressed as

$$\psi = \nu x F(\zeta) \quad T-T_\infty = \theta_p H(\zeta) \tag{7}$$

$$\zeta = J\frac{y^2}{2x^2} \quad \theta_p = \frac{Q}{2\pi\rho C_p \nu x}. \tag{8}$$

The appropriate Reynolds number R_x and Grashof number Gr_x may be defined as

$$R_x = U_\infty x/\nu \quad Gr_x = g\beta^*\theta_p x^3/\nu^2. \tag{9}$$

The two sets of the equations considered in the works [2, 3] can be obtained by employing $J = R_x$ and $J = Gr_x^{1/2}$ in the definition of the similarity variable ζ. On the other hand if $J = R_x + Gr_x^{1/2}$ is adopted a single set of equations will follow for the entire mixed convection

NOMENCLATURE

C_p	specific heat
$f(\eta)$	non-dimensional stream function defined by equation (18)
$F(\zeta)$	non-dimensional stream function defined by equation (7)
g	gravitational acceleration
Gr_x	local Grashof number defined by equation (9)
$h(\eta)$	non-dimensional temperature defined by equation (18)
$H(\zeta)$	non-dimensional temperature defined by equation (17)
J	mixed convection parameter, $(R_x^n+Gr_x^{n/2})^{1/n}$
n	an index defined by equation (10)
Pr	molecular Prandtl number
Q	heat released from the source per unit time
R_x	local Reynolds number, $U_\infty x/\nu$
T	temperature
T_∞	free-stream temperature
u	axial velocity component
v	normal velocity component
U_∞	free-stream velocity in axial direction
x	axial coordinate in vertical direction
y	normal coordinate.

Greek symbols

β	mixed convection parameter defined by relation (16) and for $n = 2$ by (24)
β^*	volumetric expansion coefficient
δ	half-width of the plume based on velocity profile defined by equation (30a)
δ_T	half-width of the plume based on temperature profile defined by equation (30b)
ζ	similarity variable defined by equation (8)
η	similarity variable, ζPr
θ_p	characteristic temperature for plume defined by equation (8)
ν	molecular kinematic viscosity
ψ	stream function.

regime. Furthermore, a more general choice

$$J = (R_x^n+Gr_x^{n/2})^{1/n} \quad n > 0 \tag{10}$$

also leads to a single set of equations.

Introducing the variables (7)–(10) in the governing equations (1)–(3) we get

$$2(\zeta F'')'+FF''+(1-\beta^n)^{2/n}H = 0 \tag{11}$$

$$2(\zeta H')'+Pr(FH)' = 0. \tag{12}$$

The boundary conditions (4) and (5) become

$$\zeta = 0, \quad F = \zeta^{1/2}F'' = \zeta^{1/2}H' = 0 \tag{13}$$

$$\zeta \to \infty \quad F' \to \beta, \quad H \to 0 \tag{14}$$

subjected to heat flux relation

$$\int_0^\infty F'H\,d\zeta = 1. \tag{15}$$

Here β is a constant, independent of x, defined as

$$\beta = R_x/(R_x^n+Gr_x^{n/2})^{1/n}. \tag{16}$$

For purely free convection flow $U_\infty = 0$, $R_x = 0$ therefore $\beta = 0$, and for purely forced convection flow the buoyancy is zero and therefore $\beta = 1$. Therefore, the two asymptotic cases of purely free and forced convection correspond to $\beta = 0$ and 1 and the mixed convection domain corresponds to $0 \leqslant \beta \leqslant 1$. The relation (16) maps the entire mixed convection domain to $0 \leqslant \beta \leqslant 1$ for all positive values of n. The number n represents the path of the mapping of the infinite domain in the Gr_x/R_x^2 plane to finite domain in the β-plane. The mapping constant n is chosen such that the single set of equations (11)–(15) in the two asymptotic cases $\beta \to 0$ and 1 directly lead to the equations [2, 3] for strongly and weakly buoyant plumes. Therefore we adopt

$$n = 2. \tag{17}$$

For large values of Pr the following transformation is appropriate,

$$\eta = \zeta\, Pr, \quad f(\eta) = Pr\, F(\zeta), \quad h(\eta) = H(\zeta)/Pr. \tag{18}$$

Based on (17) and (18) the equations (11)–(15) reduce to

$$2(\eta f'')'+Pr^{-1}ff''+(1-\beta^2)h = 0 \tag{19}$$

$$2(\eta h')'+(fh)' = 0 \tag{20}$$

$$\eta = 0, \quad f = \eta^{1/2}f'' = \eta^{1/2}h' = 0 \tag{21}$$

$$\eta \to \infty \quad f' \to \beta, \quad h \to 0 \tag{22}$$

$$\int_0^\infty f'h\,d\eta = 1. \tag{23}$$

The constant β is given by

$$\beta = (1+Gr_x/R_x^2)^{-1/2}. \tag{24}$$

For forced convection asymptote $\beta = 1$, the closed-form solution to equations (19)–(23) is [2]

$$f = \eta, \quad h = \tfrac{1}{2}e^{-\eta/2}. \tag{25}$$

For $Pr = 1$, we add equations (19) and (20) and integrate once to get

$$2f''+(1-\beta^2)h = 0. \tag{26}$$

The evaluation of integral (23) after elimination of h from (26) gives

$$f'(0) = 1 \qquad (27)$$

for all values of β. Furthermore, the elimination of h between relations (19) and (26) leads to

$$2\eta f''' + ff'' = 0 \qquad (28)$$

and the boundary conditions are

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = \beta. \qquad (29)$$

Therefore for $Pr = 1$, f may be found from solution of equation (28) subject to (29) and then h may be determined from (26).

3. RESULTS AND DISCUSSION

The equations (19) (23) have been integrated numerically for $Pr = 0.72$, 6.8, 50, 100 and ∞ for the entire mixed convection regime $0 \leqslant \beta \leqslant 1$ by the Runge–Kutta–Gill method. The equations (19)–(23) are singular at $\eta = 0$. The highest-order derivatives in these equations have been estimated at $\eta = 0$ and using these derivatives the numerical integration of equations (19)–(23) is directly started from $\eta = 0$.

The non-dimensional axial velocity distribution $\beta u/U_\infty$ is presented in Fig. 1 vs $\eta^{1/2}$ for various values of mixed convection parameter β and $Pr = 0.72$. The velocity is maximum at the axis of plume and away from axis it decreases monotonically to approach the free-stream velocity. At $\beta = 1$ the velocity distribution is that of the forced convection stream, as β decreases the velocity u/U_∞ increases due to action of buoyancy and at $\beta = 0$ the profile approaches a free convection buoyant plume. The effects of Prandtl number on centreline velocity $f'(0) = \beta u(x,0)/U_\infty$ are displayed in Fig. 2 vs mixed convection parameter β. The centreline velocity $u(x,0)/U_\infty$ always increases with β. For a fixed β the centreline velocity increases with increase of Prandtl number.

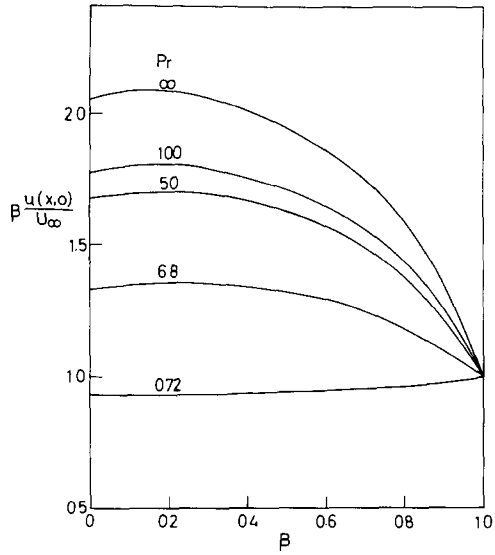

FIG. 2. Effects of Prandtl number on centreline velocity for the mixed convection buoyant plume.

The non-dimensional temperature distribution $(T-T_\infty)/\theta_p Pr$ is displayed vs $\eta^{1/2}$ in Fig. 3 for various values of mixed convection parameter β for $Pr = 0.72$. The effects of Prandtl number on centreline temperature $[T(x,0)-T_\infty]/\theta_p Pr$ are presented against mixed convection parameter β in Fig. 4. The figure shows that for $\beta \to 1$ the forced convection limit, the non-dimensional centreline temperature approaches 0.5 and as $\beta \to 0$ it approaches to the free convection value. For fixed β an increase of Prandtl number increases the centreline temperature.

The width of the plume is another important parameter. Based on velocity profile the width δ may be

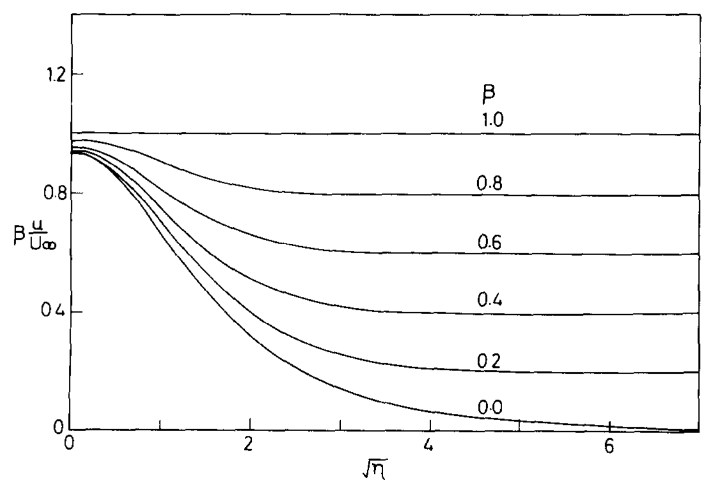

FIG. 1. The velocity profiles for mixed convection buoyant plumes in the range covering purely forced convection flow to purely free convection buoyant plume for $Pr = 0.72$.

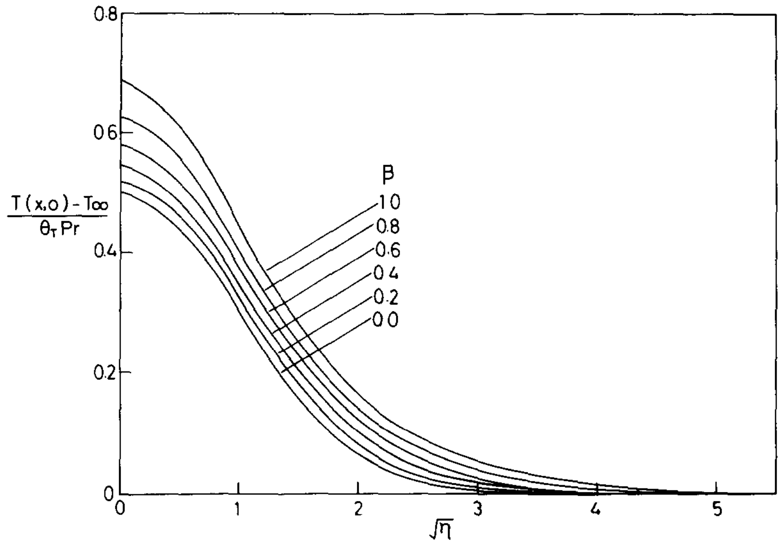

FIG. 3. The temperature profiles for mixed convection buoyant plumes in the range covering from purely forced convection to purely free convection buoyant plume for $Pr = 0.72$.

defined as the distance from the centreline where the free-stream velocity defect is a small fraction (e.g. 0.05) of the velocity defect between the axis and free stream, i.e.

$$\frac{u(x,\delta)-U_\infty}{u(x,0)-U_\infty} = 0.05. \tag{30a}$$

Likewise, the thermal width δ_T may be defined as

$$\frac{T(x,\delta_T)-T_\infty}{T(x,0)-T_\infty} = 0.05. \tag{30b}$$

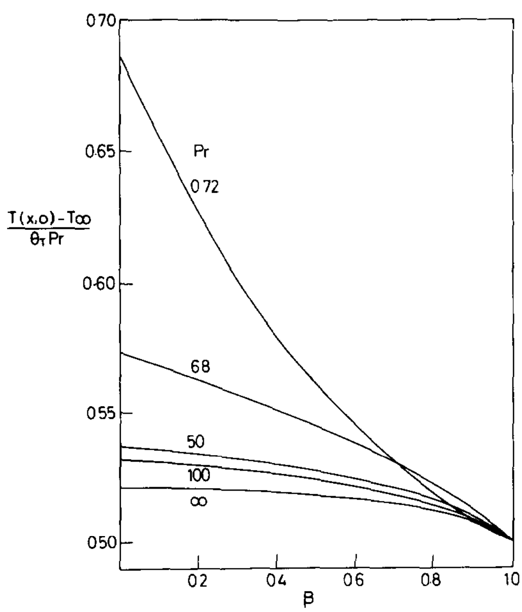

FIG. 4. Effects of Prandtl number on centreline temperature for mixed convection buoyant plume.

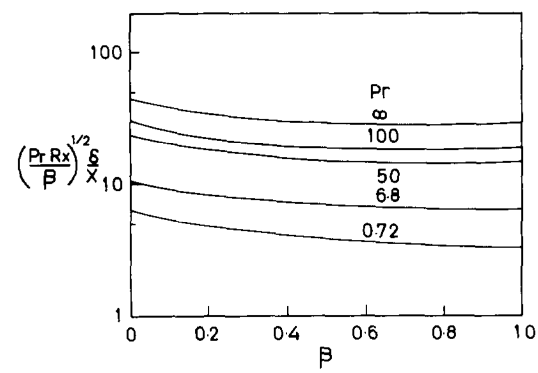

FIG. 5. Width of the plume from velocity profile for mixed convection flow.

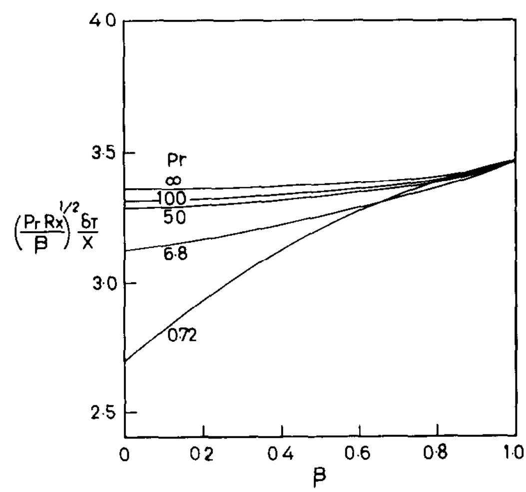

FIG. 6. Width of the plume from temperature profile for mixed convection flow.

In terms of similarity variables the expressions (30a) and (30b) reduce to

$$\frac{f'(\eta_\delta)-\beta}{f'(0)-\beta} = 0.05 \quad \frac{h(\eta_{\delta_T})}{h(0)} = 0.05 \qquad (31a,b)$$

$$2\eta_\delta = \frac{\delta^2 R_x Pr}{x^2\beta}; \quad 2\eta_{\delta_T} = \frac{\delta_T^2}{x^2}\frac{R_x Pr}{\beta}. \qquad (32a,b)$$

The non-dimensional velocity width $(\delta/x)(Pr\ R_x/\beta)^{1/2}$ determined from the velocity profiles is displayed in Fig. 5 for various values of mixed convection parameter β. The velocity width δ decreases as β decreases and for $\beta \to 0$ approaches to free convection buoyant plume. For a fixed β width δ decreases as Pr increases.

The non-dimensional thermal width $(\delta_T/x)(Pr\ R_x/\beta)^{1/2}$ is presented vs β in Fig. 6 with Pr as a parameter. Near the forced convection asymptote $\beta \to 1$ and $(\delta_T/x)(Pr\ R_x/\beta)^{1/2} \to 3.46$ a value given by the closed-form solution (25). The results for δ_T are qualitatively similar and no additional comment is needed.

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SILLAGE DE CONVECTION MIXTE AU DESSUS D'UNE SOURCE PONCTUELLE DANS UN ECOULEMENT VERTICAL

Résumé—On considère la convection mixte de couche limite au dessus d'une source ponctuelle soumise à un écoulement uniforme vertical ascendant. Dans cette étude est présentée une approche nouvelle où le domaine complet de la convection mixte est décrit par un unique système d'équations. Les solutions numériques sont présentées pour des nombre s de Prandtl 0,72; 6,8; 50; 100 et infini.

MISCHKONVEKTIONSFAHNE ÜBER EINER PUNKTFÖRMIGEN WÄRMEQUELLE IN EINER VERTIKALEN FREIEN STRÖMUNG

Zusammenfassung—Es wird die Grenzschicht bei Mischkonvektion über einer punktförmigen Wärmequelle behandelt, die gleichförmig nach oben umströmt wird. In der vorliegenden Arbeit wird eine neue Näherung vorgestellt, bei der das vollständige Gebiet der Mischkonvektion durch ein einfaches Gleichungssystem beschrieben wird. Numerische Lösungen werden für Prandtl-Zahlen von 0,72; 6,8; 50; 100 und unendlich angegeben.

СМЕШАННАЯ КОНВЕКЦИЯ НАД ТОЧЕЧНЫМ ИСТОЧНИКОМ ТЕПЛА В ВЕРТИКАЛЬНОМ СВОБОДНОМ ПОТОКЕ

Аннотация—Рассматривается пограничный слой при смешанной конвекции над точечным источником тепла, находящимся в восходящем однородном потоке. В работе представлен новый подход, при котором вся область смешанной конвекции описывается одной системой уравнений. Даны численные решения для чисел Прандтля 0,72; 6,8; 50; 100 и бесконечности.